Math 339 Test 1
Solutions

1 a) $r^3 - 3r^2 + 4r - 2 = (r-1)(r^2-2r+2) = 0$ has roots $r = 1$ and $r = \frac{+2 \pm \sqrt{4-8}}{2} = 1 \pm i$. Thus the general solution is

$$y = Ae^x + e^x(B\cos x + C\sin x)$$

b) $r^5 + 6r^4 + 12r^3 + 8r^2 = r^2(r^3 + 6r^2 + 12r + 8) = r^2(r+2)^3$, so $r = 0$ with multiplicity 2 and $r = -2$ with multiplicity 3. Thus, the general solution is

$$y = Ax + B + (Cx^2 + Dx + E)e^{-2x}$$

2a $r^4 + 4r^2 = r^2(r^2+4)$ so $r = 0$ occurs with multiplicity 2 and $r = 2i$ occurs with multiplicity 1. $e^{2x}\cos 2x$ corresponds to $\lambda = 2 \pm 2i$, which has multiplicity 0. So $y_p = e^{2x}((Ax+B)\cos 2x + (Cx+D)\sin x)$

b $r^5 - 2r^2 = r^2(r^3 - 2)$ so $r = 0$ occurs with multiplicity 2. x^3 corresponds to $\lambda = 0$, so

$$y_p = x^2(Ax^3 + Bx^2 + Cx + D)$$

c $r^4 + 16 = 0$ gives $r^2 = \pm 4i \Rightarrow r = 2\left(\pm\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}\right) = \pm\sqrt{2} \pm i\sqrt{2}$
$e^{\sqrt{2}x}\cos\sqrt{2}x$ corresponds to $\lambda = \sqrt{2} \pm i\sqrt{2}$, so this occurs with multiplicity 1. Thus $y_p = xe^{\sqrt{2}x}((Ax+B)\cos\sqrt{2}x + (Cx+D)\sin\sqrt{2}x)$

3.

$$y_1 = x \qquad y_2 = x\sin x \qquad y_3 = x\cos x$$
$$y_1' = 1 \qquad y_2' = \sin x + x\cos x \qquad y_3' = \cos x - x\sin x$$
$$y_1'' = 0 \qquad y_2'' = 2\cos x - x\sin x \qquad y_3'' = -2\sin x - \cos x$$

$$u_1' x + u_2' x\sin x + u_3' x\cos x = 0$$
$$u_1' 1 + u_2'(\sin x + x\cos x) + u_3'(\cos x - x\sin x) = 0$$
$$u_1' \cdot 0 + u_2'(2\cos x - x\sin x) + u_3'(-2\sin x - x\cos x) = x$$

$$\left(\begin{array}{ccc|c} x & x\sin x & x\cos x & 0 \\ 1 & \sin x + x\cos x & \cos x - x\sin x & 0 \\ 0 & 2\cos x - x\sin x & -2\sin x - x\cos x & x \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 0 & -x^2\cos x & x^2\sin x & 0 \\ 1 & \sin x + x\cos x & \cos x - x\sin x & 0 \\ 0 & 2\cos x - x\sin x & -2\sin x - x\cos x & x \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & \sin x + x\cos x & \cos x - x\sin x & 0 \\ 0 & 1 & -\sin x/\cos x & 0 \\ 0 & 2\cos x - x\sin x & 2\sin x - x\cos x & x \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & \cos x + \sin^2 x/\cos x & 0 \\ 0 & 1 & -\sin x/\cos x & 0 \\ 0 & 0 & -x\cos x - x\sin^2 x/\cos x & x \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1/\cos x & 0 \\ 0 & 1 & -\sin x/\cos x & 0 \\ 0 & 0 & -x/\cos x & x \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1/\cos x & 0 \\ 0 & 1 & -\sin x/\cos x & 0 \\ 0 & 0 & 1 & -\cos x \end{array}\right)$$

$$\rightarrow \begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\sin x \\ 0 & 0 & 1 & -\cos x \end{matrix}$$

$u_1' = 1$, $u_2' = -\sin x$, $u_3' = -\cos x$ $\quad$ $u_1 = x$, $u_2 = \cos x$, $u_3 = -\sin x$

$$y_p = x^2 + x\sin x\cos x - x\sin x\cos x = x^2.$$

4/ $x^2\cos 2x$ corresponds to $\lambda = 2i$ with multiplicity 3
e^x corresponds to $\lambda = 1$ with multiplicity 0
x^6 corresponds to $\lambda = 0$ with multiplicity 7

So the characteristic eq of smallest degree is

$$r^7(r-1)\left((r+2i)(r-2i)\right)^3 = r^7(r-1)\,(r^2+4)^3$$

$$= r^7(r-1)\,(r^6 + 12r^4 + 48r^2 + 64)$$

$$= r^7(r^7 - r^6 + 12r^5 - 12r^4 + 48r^3 - 48r^2 + 64r - 64)$$

Giving ODE $\quad y^{(14)} - y^{(13)} + 12y^{(12)} - 12y^{(11)} + 48y^{(10)} - 48y^{(9)} + 64y^{(8)} - 64y^{(7)} = 0$

3 (Using determinants)

$$W = \begin{vmatrix} x & x\sin x & x\cos x \\ 1 & \sin x + x\cos x & \cos x - x\sin x \\ 0 & 2\cos x - x\sin x & -2\sin x - x\cos x \end{vmatrix}$$

$$= x\left(-2\sin^2 x - 3x\cos x\sin x - x^2\cos x - \left(2\cos^2 x - 2x\sin x\cos x - x\sin x\cos x + x^2\sin^2 x\right)\right)$$

$$- \left(-2x\sin^2 x - x^2\sin x\cos x - 2x\cos^2 x + x^2\sin x\cos x\right)$$

$$= -2x\sin^2 x - 3x^2\sin x\cos x - x^3\cos x - 2x\cos^2 x + 2x^2\sin x\cos x + x^2\sin x\cos x - x^3\sin^2 x$$
$$+ 2x\sin^2 x + x^2\sin x\cos x + 2x\cos^2 x - x^2\sin x\cos x$$

$$= -x^3$$

$$W_1 = \begin{vmatrix} 0 & x\sin x & x\cos x \\ 0 & \sin x + x\cos x & \cos x - x\sin x \\ 1 & 2\cos x - x\sin x & -2\sin x - x\cos x \end{vmatrix}$$

$$= x\sin x\cos x - x^2\sin^2 x - x\sin x\cos x - x^2\cos^2 x$$

$$= -x^2$$

$$W_2 = \begin{vmatrix} x & 0 & x\cos x \\ 1 & 0 & \cos x - x\sin x \\ 0 & 1 & -2\sin x - x\cos x \end{vmatrix}$$

$$= -\left(x\cos x - x^2\sin x - x\cos x\right)$$

$$= x^2\sin x$$

$$W_3 = \begin{vmatrix} x & x\sin x & 0 \\ 1 & \sin x + x\cos x & 0 \\ 0 & 2\cos x - x\sin x & 1 \end{vmatrix}$$

$$= \quad x\sin x + x^2\cos x - x\sin x = x^2\cos x$$

$$y_p = y_1 \int \frac{W_1}{W} r(x) + y_2 \int \frac{W_2}{W} r(x) + y_3 \int \frac{W_3}{W} r(x)$$

$$= x \int \frac{-x^2}{-x^3} \cdot x \, dx + x\sin x \int \frac{x^2 \sin x}{-x^3} \cdot x \, dx + x\cos x \int \frac{x^2\cos x}{-x^3} \, x \, dx$$

$$= x \int 1 \, dx + x\sin x \int -\sin x \, dx + x\cos x \int -\cos x \, dx$$

$$= x^2 + x\sin x\cos x - x\cos \sin x =$$

$$= x^2$$